Note: Mass Assignment Theory (MAT)

A. Inoue

June 23rd, 2000

1 Mass Assignment Theory (MAT)

Mass assignment theory (MAT) is introduced by J. F. Baldwin in order to handle more semantics of data in the sense of unification operation in logic programming. Then, it is applied for qualitative summarization of numerical data by A. L. Ralescu as a replacement of statistic summarization measures such as average and median.

1.1 Basics

The basic concepts of mass assignment theory (MAT) are as follows:

Definition 1 *Let* $\mathbf{S}$ *be a sample space. A* <u>*mass assignment*</u> *(MA)* $\mathbf{m_s}$ *associated to* $\mathbf{S}$ *is a function on the power set* $\mathcal{P}(S)$ *over an interval of real numbers* $m_s : \mathcal{P}(S) \mapsto [0, 1]$ *such that*

$$\sum_{A \subseteq S} m_s(A) = 1 \tag{1}$$

Definition 2 *A subset* $\mathbf{A}$ *of a sample space* $\mathbf{S}$ *is called a* <u>*focal element*</u> *if*

$$m_s(A) > 0 \tag{2}$$

Example 1 *Let* $\mathbf{S} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \mathbf{x}\}$*. Then a function* $\mathbf{m} : \mathcal{P}(\mathbf{S}) \mapsto [\mathbf{0}, \mathbf{1}]$ *defined by*

$$m(A) = \begin{cases} 0.6 & A = \{a, b\} \\ 0.4 & A = \{b, c\} \\ 0 & otherwise \end{cases}$$

is a MA on $\mathbf{S}$ *with focal elements* $\{\mathbf{a}, \mathbf{b}\}$ *and* $\{\mathbf{b}, \mathbf{c}\}$*.*

Comparing the above definition with that of basic probability assignment in Dempster-Shafer theory, note that it is not required that

$$m(\emptyset) = 0$$

The MA $\mathbf{m}$ is said to be <u>complete</u> if $m(\emptyset) = 0$ and <u>incomplete</u> if $m(\emptyset) \neq 0$.

MAT departs from Dempster-Shafer theory of evidence in that it provides a full calculus of mass assignments as opposed to only the combination rule provided by Dempster-Shafer theory, and the calculus beyond the verification role is enhanced. As a result, MAT furnish the calculus to handle imprecision whereas Dempter-Shafer theory of evidence deals mainly with uncertainty caused by lack of information from probability point of view.

1.2 Mass Assignment Theory and Probability

In MAT, there exists the following relation between a discrete probability distribution, e.g. a normalized histogram, associated to elements of a sample space $\mathbf{S}$ and a given MA $\mathbf{m_s}$ on $\mathbf{S}$:

Given a discrete probability distribution P_s on a sample space S, a MA m_s on the power set $\mathcal{P}(S)$, and a probability distribution P_A for each $A \in \mathcal{P}(S)$, the following equation holds.

$$P_s(x) = \sum_{A \subseteq S, x \in A} P_A(x) \cdot m_s(A) \tag{3}$$

For convenience, we call $\mathbf{P_A(x)}$ a <u>selection rule</u> (denoted by Ralescu).

Example 2 *Let* $P_{\{a,b\}}(a) = P_{\{a,b\}}(b) = 0.5$ *and* $P_{\{b,c\}}(b) = P_{\{b,c\}}(c) = 0.5$ *be selection rules in Example 1 above. Then using (3), we calculate the probabilities of a, b, and c as:*

$$\begin{aligned} P_S(a) &= P_{\{a,b\}}(a) \times m(\{a,b\}) = 0.5 \times 0.6 = 0.3 \\ P_S(b) &= P_{\{a,b\}}(b) \times m(\{a,b\}) + P_{\{b,c\}}(b) \times m(\{b,c\}) \\ &= 0.5 \times 0.6 + 0.5 \times 0.4 = 0.5 \\ P_S(c) &= P_{\{b,c\}}(c) \times m(\{b,c\}) = 0.5 \times 0.4 = 0.2 \end{aligned}$$

The selection rules in Example 2 correspond to a uniform distribution hence referred to as the *least prejudged distribution*(LPD). Alternatively, the selection rule which concentrates entire distribution on only one value in a focal element, and called the *most prejudged distribution*(MPD), will yield a different probability distributions for example, in Example 2, the selection rules $P_{\{a,b\}}(a) = 1.0$ *and* $P_{\{a,b\}}(b) = 0.0$ are MPD because a has its entire distribution, and it makes the probability distribution different from the one with LPD as follows (it is assumed that other selection rules remain the same):

$$\begin{aligned} P_S(a) = 0.6 \\ P_S(b) = 0.2 \\ P_S(c) = 0.2 \end{aligned}$$

As it can be seen in above examples, different selection rules will yield different overall distributions. This plays the key role of managing biases for Perceptual Information Processing.

1.3 Mass Assignment Theory and Fuzzy Sets

Using the representation theorem it can now be stated that a fuzzy set corresponds to a special mass assignment. Indeed, let **S** be a universe of discourse, $S = \{x_1, \ldots, x_n\}$, and **F** a normal fuzzy set defined on **S**, Then

$$F = x_1/\mu_1 + x_2/\mu_2 + \ldots + x_n/\mu_n$$

where without loss of generality we assume

$$1 = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > \mu_{n+1} = 0 \tag{4}$$

Then

$$m_f(A) = \begin{cases} \mu_i - \mu_{i+1} & \text{if } A = \{x_1 \ldots xi\} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

is a mass assignment (MA) on **S** with nested focal elements $\{x_1 \ldots xi\}$, $i = 1 \ldots n$.

It is easy to show that

$$m_f(A) \geq 0 \text{ if } A = \{x_1 \ldots x_i\} \tag{6}$$

and that

$$\sum_{i=1}^{n} m_f(\{x_1, \ldots, x_i\}) = 1 \tag{7}$$

Example 3 *Let* $S = \{x_1, x_2, x_3\}$ *and* **f** *be a fuzzy set defined as*

$$f = x_1/1 + x_2/0.7 + x_3/0.5$$

Then

$$\begin{aligned} m_f(\{x_1\}) &= 0.3 \\ m_f(\{x_1, x_2\}) &= 0.2 \\ m_f(\{x_1, x_2, x_3\}) &= 0.5 \end{aligned} \tag{8}$$

is a MA. A fuzzy set **f** *based on this relation is computed as follows:*

$$\begin{array}{lrcl} x_1 : & \mu_1 & = & 1 \\ x_2 : & m_f(\{x_1\}) & = & \mu_1 - \mu_2 = 0.3 \Rightarrow \mu_2 = 0.7 \\ x_3 : & m_f(\{x_1, x_2\}) & = & \mu_2 - \mu_3 = 0.2 \Rightarrow \mu_3 = 0.5 \end{array} \tag{9}$$

It can be seen that given a mass assignment with nested focal elements we obtain a normal fuzzy set, i.e. a fuzzy set whose maximum membership value is 1.0, with membership function related to the mass assignment by (5). Indeed, for Example 3, if we start with the mass assignment defined by (8), we obtain the membership function (9).

1.4 Probability and Fuzzy Sets

Using the relation between mass assignments and fuzzy sets and mass assignments and probabilities we can now obtain the mapping between probability and a fuzzy set as shown in Figure 1. Let $P_s(x_k)$ be a probability of a sample space $\mathbf{S}$, and $P_{A_i}(x_k)$ be a selection rule for $\mathbf{x_k}$ from the focal element $A_i = x_1 \ldots x_i, i = 1, \ldots k$ of a mass assignment. Then

$$P_s(x_k) = \sum_{i=k}^{n} P_{A_i}(x_k) \cdot (\mu_i - \mu_{i+1}) \qquad (10)$$

We note that all focal elements are nested as they correspond to the level sets (α-cuts) for $\mu_i \forall i = 1, \ldots, n$.

The main role of the selection rules is in maintaining consistency between a fuzzy set and different probability distributions which satisfy (3). The selection rules P_{A_i} can be tuned if the fuzzy set, (i.e. the membership values μ_i's) is always manually changed in order for P_S to remain the same. This feature is important to determine the valid range of data for a given fuzzy set. Inconsistency in the data set is detected by obtaining some invalid probability in(10). Such results are obtained when the order of membership values assumed in (4) is not maintained. From a different angle, this can also be used to determine what is lacking in order to keep the consistency.

As follows from (10) and the previous discussion the selection rules can be used to establish a many-to-many relationship between probability distributions of data and its fuzzy set definitions. Selection rules can also be one way of implementing user's perception. In this case, selection rules are given arbitrarily. Then either a fuzzy set for a given data set or an ideal data set biased by user's perception (i.e. selection rules) for a fuzzy set representing a concept can be obtained by (10) (see Figure 1).

2 Operations of Mass Assignments

One of the attractive features of MAT is that operations of MA are defined in a way compatible to set operations. They include the complement ($\tilde{}$), meet ($\wedge$), and join ($\vee$). The complement is determined uniquely. However, the join and meet operations are not determined uniquely because of possible combinations of redistribution of mass over new focal elements determined by taking either intersection (meet) or union (join) of original focal elements. The general definitions of these operations are as follows:

Let

$$m = \{M_i : m_i\}$$

and

$$n = \{N_i : n_i\}$$

be two mass assignments on universal set X. The meet $m \wedge n$ is a mass assignment

$$m \wedge n = \{L_k : l_k\}$$

where

$$L_k = M_i \cap N_j$$

and

$$l_k = \sum_{i,j\ ;\ L_{ij}=L_k} l_{ij} \tag{11}$$

$$\sum_j l_{ij} = m_i \ \forall\ i \tag{12}$$

$$\sum_i l_{ij} = n_j \ \forall\ j \tag{13}$$

The join operation is defined in a similar way by equations (11), (12), and (13) with the difference that

$$L_k = M_i \cup N_j$$

Equations (12) and (13) are referred to as the row and column constraints respectively.

Example 4 illustrates the meet and the join of two mass assignments as follows:

Example 4 *Mass assignments* $\mathbf{m_1}$ *and* $\mathbf{m_2}$ *with respect to universal set* $\mathbf{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *are given as follows:*

$$\begin{aligned} m_1(\{a\}) &= 0.2 \\ m_1(\{a,b\}) &= 0.5 \\ m_1(\{b,c\}) &= 0.3 \end{aligned}$$

$$\begin{aligned} m_2(\{b\}) &= 0.3 \\ m_2(\{a,c\}) &= 0.4 \\ m_2(\{a,b,c\}) &= 0.3 \end{aligned}$$

2.1 Complement

The complement of m_1, $\overline{\mathbf{m_1}}$ is defined as follows:

$$\overline{m_1}(A) = m_1(\overline{A}) \text{ for } A \in \mathcal{P}(S) \tag{14}$$

For our example, we obtain

$$\begin{array}{lclclcl} \overline{m_1}(\{b,c\}) &=& m_1(\overline{\{b,c\}}) &=& m_1(\{a\}) &=& 0.2 \\ \overline{m_1}(\{c\}) &=& m_1(\overline{\{c\}}) &=& m_1(\{a,b\}) &=& 0.5 \\ \overline{m_1}(\{a\}) &=& m_1(\overline{\{a\}}) &=& m_1(\{b,c\}) &=& 0.3 \end{array}$$

Note that the focal elements of $\overline{\mathbf{m_1}}$ are the complements of the focal elements of $\mathbf{m_1}$.

2.2 Meet

The meet operation computes the intersection of two mass assignments, namely $\mathbf{m_1}$ and $\mathbf{m_2}$. The name, meet, is brought from the operation for relational database. Table 1 shows redistribution of mass over set intersections of each focal element of $\mathbf{m_1}$ with each focal element of $\mathbf{m_2}$ under the constraints that column and raw sums must be equal to the corresponding masses. Therefore, the redistribution of mass assignments is not unique, and Table 1 shows one way of generating such redistribution by multiplication.

The final form for the meet $\mathbf{m_1} \wedge \mathbf{m_2}$ is obtained by taking the sum of redistributed mass whose focal element is identical as follows:

$$\begin{aligned}
(m_1 \wedge m_2)(\{a\}) &= 0.34 \\
(m_1 \wedge m_2)(\{b\}) &= 0.24 \\
(m_1 \wedge m_2)(\{c\}) &= 0.12 \\
(m_1 \wedge m_2)(\{a,b\}) &= 0.15 \\
(m_1 \wedge m_2)(\{b,c\}) &= 0.09 \\
(m_1 \wedge m_2)(\emptyset) &= 0.06
\end{aligned}$$

An alternative redistribution of mass over the same set intersection, such that masses are redistributed over the diagonal cells or those as near the diagonal as possible is shown in Table 2 (called maximum algorithm): Meet $\mathbf{m_1} \wedge \mathbf{m_2}$, then, is obtained as follows:

$$\begin{aligned}
(m_1 \wedge m_2)(\{a\}) &= 0.4 \\
(m_1 \wedge m_2)(\{b\}) &= 0.1 \\
(m_1 \wedge m_2)(\{b,c\}) &= 0.3 \\
(m_1 \wedge m_2)(\emptyset) &= 0.2
\end{aligned}$$

2.3 Join

The join operation computes the union of two mass assignments $\mathbf{m_1}$ and $\mathbf{m_2}$ and is obtained in a way similar to the meet operation. Like the meet operation the join is inherited from the corresponding operations in relational database models. Table 3 shows redistribution of mass over set union of each focal element of $\mathbf{m_1}$ with each focal element of $\mathbf{m_2}$ under row and column constraints (as previously described for the meet operation). Therefore, the redistribution of mass assignments for the join is not unique either.

Like for the meet, the join $\mathbf{m_1} \vee \mathbf{m_2}$ is obtained by taking sum of redistributed mass whose focal element is identical as follows:

$$\begin{aligned}
(m_1 \vee m_2)(\{a,b\}) &= 0.21 \\
(m_1 \vee m_2)(\{a,c\}) &= 0.08 \\
(m_1 \vee m_2)(\{b,c\}) &= 0.09 \\
(m_1 \vee m_2)(\{a,b,c\}) &= 0.62
\end{aligned}$$

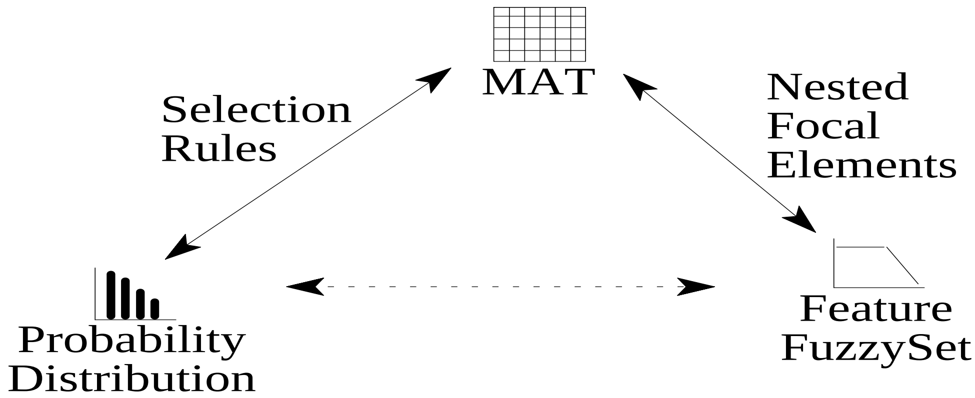

Figure 1: Relation betweenData, Fuzzy Set, and MAT

Table 1: Meet Operation by Multiplication

$m_1 \wedge m_2$	$\{b\} : 0.3$	$\{a, c\} : 0.4$	$\{a, b, c\} : 0.3$
$\{a\} : 0.2$	$\phi : 0.06$	{a}:0.08	{a}:0.06
$\{a, b\} : 0.5$	$\{b\} : 0.15$	$\{a\} : 0.2$	$\{a, b\} : 0.15$
$\{b, c\} : 0.3$	$\{b\} : 0.09$	$\{c\} : 0.12$	$\{b, c\} : 0.09$

Table 2: Alternate Meet Operation (Maximum Algorithm)

$m_1 \wedge m_2$	$\{b\} : 0.3$	$\{a, c\} : 0.4$	$\{a, b, c\} : 0.3$
$\{a\} : 0.2$	$\phi : 0.2$	$\{a\} : 0.0$	$\{a\} : 0.0$
$\{a, b\} : 0.5$	$\{b\} : 0.1$	$\{a\} : 0.4$	$\{a, b\} : 0.0$
$\{b, c\} : 0.3$	$\{b\} : 0.0$	$\{c\} : 0.0$	$\{b, c\} : 0.3$

Table 3: Join Operation by Multiplication

$m_1 \vee m_2$	$\{b\} : 0.3$	$\{a, c\} : 0.4$	$\{a, b, c\} : 0.3$
$\{a\} : 0.2$	$\{a, b\} : 0.06$	$\{a, c\} : 0.08$	$\{a, b, c\} : 0.06$
$\{a, b\} : 0.5$	$\{a, b\} : 0.15$	$\{a, b, c\} : 0.2$	$\{a, b, c\} : 0.15$
$\{b, c\} : 0.3$	$\{b, c\} : 0.09$	$\{a, b, c\} : 0.12$	$\{a, b, c\} : 0.09$

The maximum algorithm alternative redistribution of mass over the same set union is shown in Table 4. In this case, the join $\mathbf{m_1} \vee \mathbf{m_2}$ is obtained as:

$$\begin{aligned}(m_1 \vee m_2)(\{a,b\}) &= 0.3 \\ (m_1 \vee m_2)(\{a,b,c\}) &= 0.7\end{aligned}$$

2.4 Properties of MAT Operations

It can be shown that the meet, union, and negation of mass assignment satisfy De Morgan's law such that

$$\overline{m_1 \wedge m_2} = \overline{m_1} \vee \overline{m_2} \tag{15}$$

The relations between mass assignments and fuzzy sets are consistent at the level of operations. More precisely the multiplication version of meet and join operations are equivalent to algebraic intersection and union fuzzy sets operations, and those given by maximum algorithm are equivalent to the standard fuzzy set operations.

3 Semantic Unification

Semantic unification is used to compare two fuzzy sets. The result is given as either a support pair, an interval of a probability, or a single valued probability (it is within the interval, of course). FRIL, a logic programming language with an extension of handling fuzzy sets as its terms, determines the truth value of Hone closures containing fuzzy sets based on the semantic unification. Please note that, unlike a symbolic unification, the semantic unification m between fuzzy sets g and g' is not symmetric, i.e.

$$m_{g|g'} \neq m_{g'|g} \tag{16}$$

3.1 Multiplication model

Let g and g' be fuzzy sets defined on X. The mass assignment on the truth set of g given g', conditional mass assignment, denoted by $m_{g|g'}$ defined over $\{t, f\}$ where t represents true and f represents false is determined such that

$$m_{g|g'} = \begin{cases} t: & \sum_{T(L_i|M_j)=t} l_i \cdot m_j \quad (= n_t) \\ u: & \sum_{T(L_i|M_j)=u} l_i \cdot m_j \quad (= n_u) \\ f: & \sum_{T(L_i|M_j)=f} l_i \cdot m_j \quad (= n_f) \end{cases} \tag{17}$$

where

$$T(L_i|M_j) = \begin{cases} t & if\ M_j \subseteq L_i \ \&\ M_j \neq \emptyset \\ f & if\ M_j \cap L_i = \emptyset \ \&\ M_j \neq \emptyset \\ u & otherwise \end{cases}$$

and mass assignments corresponding to fuzzy sets g and g' are defined as

$$m_g = \{L_i \ : \ l_i\}$$

and

$$m_{g'} = \{M_i \ : \ m_i\}$$

respectively. The *support pair* $g|g'$ is given by

$$g|g' = [n_t, n_t + n_u] \tag{18}$$

based on $m_{g|g'}$.

Let, for example, fuzzy sets g and g' be

$$g = a/1 + b/0.7 + c/0.2$$

and

$$g' = a/0.2 + b/1 + c/0.7 + d/0.1$$

defined on $X = \{a, b, c, d, e\}$ so that the corresponding mass assignments become

$$m_g = \{a\} : 0.3,\ \{a, b\} : 0.5,\ \{a, b, c\} : 0.2$$

and

$$m'_g = \{b\} : 0.3,\ \{b, c\} : 0.5,\ \{a, b, c\} : 0.1,\ \{a, b, c, d\} : 0.1$$

respectively. Using the tabular representation (used for meet and join) shown in Table3.1, the conditional mass assignment is computed such that

$$m_{g|g'} = \begin{cases} t : 0.33 \\ f : 0.24 \\ u : 0.43 \end{cases}$$

The support pair $g|g'$ thus becomes

$$g|g' = [0.33, 0.76]$$

3.2 Semantic Unification by Maximum Algorithm

There is an alternate computation of conditional mass assignment $m_{g|g'}$ such that each mass in the tabular representation n_{ij} is determined arbitrarily with the constraints

$$\sum_j n_{ij} = l_i$$

and

$$\sum_i n_{ij} = m_j$$

where $m_g = \{L_i : l_i\}$ and $m_{g'} = \{M_j : m_j\}$. One of results of the former example using the alternate computation becomes

$$g|g' = [0.1, 0.7]$$

as shown in Table 3.2. Please note that this is identical to the maximum algorithm used in mass assignment operations. Therefore, solutions are not determined uniquely. Note also that an interpretation of the result can be given based on the voting model as shown in page 79 of FRIL book. Each solution corresponds to a unique voting pattern.

3.3 Point Value Semantic Unification

The results of semantic unification can be determined as a single value such that

$$n_{ij} = \frac{|L_i \cap M_j|}{|M_j|} \cdot l_i \cdot m_j \tag{19}$$

The point probability is given by

$$P(g|g') = \sum_{i,j} n_{ij}$$

It is important to note that

$$P(g|g') + P(\overline{g}|g') = 1.0$$

is held on point value semantic unification.

The result of point value semantic unification for the former example is

$$P(g|g') = 0.53908$$

as shown in Table 3.3.

3.4 Possibilistic Support Pair

Possibilistic support pair can be defined in analogous to probabilistic support pair such that

$$g|g' = [\pi(g|g'), \Pi(g|g')] \tag{20}$$

where the possibility measure is given as

$$\Pi = MAX(g \cap g')$$

and the necessity measure is given as

$$\pi(g|g') = 1 - \Pi(\overline{g}|g')$$

The possibilistic support pair for the former example becomes

$$g|g' = [0.3, 0.7]$$

as $\Pi(g|g') = 0.7$ and $\pi(g|g') = 0.3$ (since $\Pi(\overline{g} \cap g') = 0.7$). This solution can be identified as a solution of the maximum algorithm as shown in Table 3.4.

Table 4: Alternative Join Operation by Maximum Algorithm

$m_1 \vee m_2$	$\{b\} : 0.3$	$\{a,c\} : 0.4$	$\{a,b,c\} : 0.3$
$\{a\} : 0.2$	$\{a,b\} : 0.2$	$\{a,c\} : 0.0$	$\{a,b,c\} : 0.0$
$\{a,b\} : 0.5$	$\{a,b\} : 0.1$	$\{a,b,c\} : 0.4$	$\{a,b,c\} : 0.0$
$\{b,c\} : 0.3$	$\{b,c\} : 0.0$	$\{a,b,c\} : 0.0$	$\{a,b,c\} : 0.3$

$L_i \backslash M_i$	$\{b\} : 0.3$	$\{b,c\} : 0.5$	$\{a,b,c\} : 0.1$	$\{a,b,c,d\} : 0.1$
$\{a\} : 0.3$	$f : 0.09$	$f : 0.15$	$u : 0.03$	$u : 0.03$
$\{a,b\} : 0.5$	$t : 0.15$	$u : 0.25$	$u : 0.05$	$u : 0.05$
$\{a,b,c\} : 0.2$	$t : 0.06$	$t : 0.1$	$t : 0.02$	$u : 0.02$

Table 5: Tabular Representation of $m_{g|g'}$

$L_i \backslash M_i$	$\{b\} : 0.3$	$\{b,c\} : 0.5$	$\{a,b,c\} : 0.1$	$\{a,b,c,d\} : 0.1$
$\{a\} : 0.3$	$f : 0.3$	$f : 0$	$u : 0$	$u : 0$
$\{a,b\} : 0.5$	$t : 0$	$u : 0.5$	$u : 0$	$u : 0$
$\{a,b,c\} : 0.2$	$t : 0$	$t : 0$	$t : 0.1$	$u : 0.1$

Table 6: Tabular Representation of $m_{g|g'}$ by Maximum Algorithm

$L_i \backslash M_i$	$\{b\} : 0.3$	$\{b,c\} : 0.5$	$\{a,b,c\} : 0.1$	$\{a,b,c,d\} : 0.1$
$\{a\} : 0.3$	$f : 0$	$f : 0$	$u : 0.01$	$u : 0.00075$
$\{a,b\} : 0.5$	$t : 0.15$	$u : 0.125$	$u : 0.03333$	$u : 0.025$
$\{a,b,c\} : 0.2$	$t : 0.06$	$t : 0.1$	$t : 0.02$	$u : 0.015$

Table 7: Tabular Representation of Point Value Semantic Unification

$L_i \backslash M_i$	$\{b\} : 0.3$	$\{b,c\} : 0.5$	$\{a,b,c\} : 0.1$	$\{a,b,c,d\} : 0.1$
$\{a\} : 0.3$	$f : 0.2$	$f : 0.1$	$u : 0$	$u : 0$
$\{a,b\} : 0.5$	$t : 0.1$	$u : 0.2$	$u : 0.1$	$u : 0.1$
$\{a,b,c\} : 0.2$	$t : 0$	$t : 0.2$	$t : 0$	$u : 0$

Table 8: Tabular Representation of Conditional Possibility Measure

3.5 Semantic Unification and Index of Intersection

Both semantic unification and index of intersection are used to compare fuzzy sets. There are similarities on their computations. At this moment, it is an opened problem to find such relations. Unfortunately, the middle exclusion law does not hold in fuzzy sets and because of this

$$II(g|g') + II(\overline{g}|g') \geq 1.0$$

where

$$II(g|g') = \frac{|g \cap g'|}{|g'|} \tag{21}$$

and sigma-count cardinality is used in this case. On the other hand, we have the middle exclusion law on the point value semantic unification.